



The Open
University

MST125

Essential mathematics 2

Exercise Booklet 11

1 Eigenvalues and eigenvectors of matrices

Exercise 1

- (a) Let $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$. Show that if $\mathbf{x} = \begin{pmatrix} k \\ 0 \end{pmatrix}$ for some number k , then $\mathbf{Ax} = 4\mathbf{x}$.
- (b) Let $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Show that if $\mathbf{x} = \begin{pmatrix} k \\ k \end{pmatrix}$ for some number k , then $\mathbf{Ax} = 5\mathbf{x}$.

Exercise 2

Show that the vectors $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix},$$

and find the corresponding eigenvalues.

Exercise 3

For each of the following matrices, find the trace and determinant and hence find the characteristic equation.

- (a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & -7 \end{pmatrix}$
- (c) $\mathbf{A} = \begin{pmatrix} -3 & 0 \\ 4 & -5 \end{pmatrix}$

Exercise 4

Find the eigenvalues of the following matrices.

- (a) $\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}$
- (c) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 6 & 2 \end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix} -2 & 3 \\ -3 & -2 \end{pmatrix}$

Exercise 5

The matrix $\begin{pmatrix} 1 & 4 \\ 9 & 1 \end{pmatrix}$ has eigenvalues -5 and 7 . For each of these eigenvalues, find a corresponding eigenvector.

Exercise 6

Find the eigenvalues of the following matrices, and for each eigenvalue find a corresponding eigenvector.

- (a) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -6 & -10 \end{pmatrix}$
- (c) $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$ (d) $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

Exercise 7

Show that the vectors

$$\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

are eigenvectors of the matrix

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix},$$

and for each eigenvector find the corresponding eigenvalue.

Exercise 8

In this exercise you can practise some of the ideas about mathematical language and proof that you learned in Unit 9, by proving some results about eigenvalues and eigenvectors.

Recall that an *eigenvalue* of a square matrix $\mathbf{A}$ is defined to be a number λ such that

$$\mathbf{Ax} = \lambda\mathbf{x}$$

for some non-zero vector $\mathbf{x}$. This non-zero vector $\mathbf{x}$ is called an *eigenvector* of $\mathbf{A}$ corresponding to λ .

Use this definition to prove each of the following for a square matrix $\mathbf{A}$.

- (a) If λ is an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{x}$, then λ^n is an eigenvalue of $\mathbf{A}^n$ with the same corresponding eigenvector $\mathbf{x}$, for all $n \in \mathbb{N}$.

Hint: use mathematical induction.

- (b) If λ is an eigenvalue of $\mathbf{A}$, then:
- (i) $-\lambda$ is an eigenvalue of $-\mathbf{A}$;
 - (ii) $\lambda + 1$ is an eigenvalue of $\mathbf{A} + \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.
- (c) If $\mathbf{A}^2 = \mathbf{A}$ and λ is an eigenvalue of $\mathbf{A}$, then $\lambda = 0$ or $\lambda = 1$.
- (d) If $\mathbf{A}$ is invertible and λ is an eigenvalue of $\mathbf{A}$, then $\lambda \neq 0$.

Hint: use proof by contradiction.

2 Eigenvalues and eigenvectors of special types of matrices

Exercise 9

Find the eigenvalues of the following matrices, and for each eigenvalue find a corresponding eigenvector.

- (a) $\begin{pmatrix} -2 & 0 \\ 4 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 \\ 2 & -3 \end{pmatrix}$
- (c) $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$

Exercise 10

Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 0 \\ 9 & 1 \end{pmatrix},$$

which represents a vertical shear. For each eigenvalue find a corresponding eigenvector.

Exercise 11

Find the eigenvalues of the following matrices, each of which represents a flattening. For each eigenvalue find a corresponding eigenvector.

(a) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$

Exercise 12

Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which represents a rotation.

Exercise 13

Using the facts that you have seen in Subsection 2.3 of Unit 11 about the eigenvalues and eigenvectors of a matrix that represents a reflection, find the eigenvalues of the following matrices representing reflections, and find a corresponding eigenvector for each eigenvalue.

(a) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$

Exercise 14

Let f be the linear transformation represented by the matrix $\mathbf{A}$ with eigenvalues 1 and 5, and corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, respectively. Without finding $\mathbf{A}$, find the images of the following points under f .

- (a) (3, 3) (b) (-2, 2) (c) (7, 1)

Hint for part (c): $\begin{pmatrix} 7 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

3 Diagonalising matrices

Exercise 15

Diagonalise the following matrices.

(a) $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix}$

(c) $\mathbf{A} = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}$

Exercise 16

Find the following powers of matrices.

(a) $\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}^3$ (b) $\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}^4$

(c) $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}^{10}$

Exercise 17

(a) Diagonalise the matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}$,
and hence find $\mathbf{A}^6$.

(b) Diagonalise the matrix $\mathbf{A} = \begin{pmatrix} 5 & 9 \\ -3 & -7 \end{pmatrix}$,
and hence find $\mathbf{A}^3$.

Exercise 18

Use the model found in Subsection 3.3 of Unit 11 to predict the populations of wolves and hares after 10 years given that there are initially 500 wolves and 2000 hares.

4 Systems of differential equations

Exercise 19

Write the following systems of differential equations in matrix notation.

(a) $\dot{x} = x + 2y$ (b) $\dot{x} = 10y$
 $\dot{y} = 3x + 4y$ $\dot{y} = -x + y$

Exercise 20

Find the general solutions of the following systems of differential equations.

(a) $\dot{x} = 5x$ (b) $\dot{x} = -0.1x$
 $\dot{y} = -4y$ $\dot{y} = 0.6y$

Exercise 21

Find the general solutions of the following systems of differential equations.

(a) $\dot{x} = 3x + 2y$ (b) $\dot{x} = -13x + 5y$
 $\dot{y} = x + 4y$ $\dot{y} = -30x + 12y$

Solutions to exercises

Solution to Exercise 1

(a) We have

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} k \\ 0 \end{pmatrix} = \begin{pmatrix} 4k \\ 0 \end{pmatrix} = 4 \begin{pmatrix} k \\ 0 \end{pmatrix}.$$

Hence $\mathbf{Ax} = 4\mathbf{x}$.

(b) We have

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} k \\ k \end{pmatrix} = \begin{pmatrix} 5k \\ 5k \end{pmatrix} = 5 \begin{pmatrix} k \\ k \end{pmatrix}.$$

Hence $\mathbf{Ax} = 5\mathbf{x}$.

Solution to Exercise 2

We have

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix} = 4 \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

so $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 4.

Also,

$$\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = - \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue -1 .

Solution to Exercise 3

(a) We have

$$\text{tr } \mathbf{A} = 1 + 4 = 5,$$

$$\det \mathbf{A} = 1 \times 4 - 2 \times 3 = -2.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 5\lambda - 2 = 0.$$

(b) We have

$$\text{tr } \mathbf{A} = 1 + (-7) = -6,$$

$$\det \mathbf{A} = 1 \times (-7) - (-2) \times 3 = -1.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 6\lambda - 1 = 0.$$

(c) We have

$$\text{tr } \mathbf{A} = (-3) + (-5) = -8,$$

$$\det \mathbf{A} = (-3) \times (-5) - 0 \times 4 = 15.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 8\lambda + 15 = 0.$$

Solution to Exercise 4

(a) We have

$$\text{tr } \mathbf{A} = 2 + 3 = 5,$$

$$\det \mathbf{A} = 2 \times 3 - 6 \times 2 = -6.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 5\lambda - 6 = 0.$$

Hence

$$(\lambda + 1)(\lambda - 6) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -1 and 6 .

(Check: $(-1) + 6 = 5 = \text{tr } \mathbf{A}$.)

(b) We have

$$\text{tr } \mathbf{A} = 2 + 2 = 4,$$

$$\det \mathbf{A} = 2 \times 2 - 5 \times 0 = 4.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 4\lambda + 4 = 0.$$

Hence

$$(\lambda - 2)^2 = 0,$$

so $\mathbf{A}$ has a single, repeated eigenvalue 2 .

(Check: $2 + 2 = 4 = \text{tr } \mathbf{A}$.)

(c) We have

$$\text{tr } \mathbf{A} = 2 + 2 = 4,$$

$$\det \mathbf{A} = 2 \times 2 - 1 \times 6 = -2.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 4\lambda - 2 = 0.$$

Completing the square on the left-hand side gives

$$(\lambda - 2)^2 - 6 = 0,$$

that is,

$$(\lambda - 2)^2 = 6.$$

Taking square roots of both sides gives

$$\lambda - 2 = \pm\sqrt{6},$$

so

$$\lambda = 2 \pm \sqrt{6}.$$

Therefore the eigenvalues of $\mathbf{A}$ are

$2 - \sqrt{6}$ and $2 + \sqrt{6}$.

(Check: $(2 - \sqrt{6}) + (2 + \sqrt{6}) = 4 = \text{tr } \mathbf{A}$.)

Exercise Booklet 11

(d) We have

$$\begin{aligned}\operatorname{tr} \mathbf{A} &= (-2) + (-2) = -4, \\ \det \mathbf{A} &= (-2) \times (-2) - 3 \times (-3) = 13.\end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 4\lambda + 13 = 0.$$

Completing the square on the left-hand side gives

$$(\lambda + 2)^2 + 9 = 0,$$

that is,

$$(\lambda + 2)^2 = -9.$$

Taking square roots of both sides gives

$$\lambda + 2 = \pm 3i,$$

so

$$\lambda = -2 \pm 3i.$$

Therefore the eigenvalues of $\mathbf{A}$ are $-2 - 3i$ and $-2 + 3i$.

(Check:

$$(-2 - 3i) + (-2 + 3i) = -4 = \operatorname{tr} \mathbf{A}.)$$

(You might have preferred to solve the characteristic equations in parts (c) and (d) of this exercise by using the quadratic formula.)

Solution to Exercise 5

The eigenvector equations have the form

$$\begin{pmatrix} 1 - \lambda & 4 \\ 9 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -5$, we obtain

$$\begin{pmatrix} 6 & 4 \\ 9 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned}6x + 4y &= 0, \\ 9x + 6y &= 0.\end{aligned}$$

This pair of equations is equivalent to the single equation

$$3x + 2y = 0, \quad \text{that is,} \quad 2y = -3x.$$

If $x = 2$, then $2y = -3 \times 2$, so $y = -3$. Hence

$\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is an eigenvector of the matrix

corresponding to the eigenvalue -5 .

(Check:

$$\begin{pmatrix} 1 & 4 \\ 9 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -10 \\ 15 \end{pmatrix} = -5 \begin{pmatrix} 2 \\ -3 \end{pmatrix}.)$$

When $\lambda = 7$, the eigenvector equation is

$$\begin{pmatrix} -6 & 4 \\ 9 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned}-6x + 4y &= 0, \\ 9x - 6y &= 0.\end{aligned}$$

This pair of equations is equivalent to the single equation

$$3x - 2y = 0, \quad \text{that is,} \quad 2y = 3x.$$

If $x = 2$, then $2y = 3 \times 2$, so $y = 3$. Hence

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector of the matrix

corresponding to the eigenvalue 7.

$$(\text{Check: } \begin{pmatrix} 1 & 4 \\ 9 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 21 \end{pmatrix} = 7 \begin{pmatrix} 2 \\ 3 \end{pmatrix}.)$$

(Throughout these exercises you may obtain eigenvectors different to those given in the solutions. However, each eigenvector that you obtain should be a scalar multiple of the one given in the solution, unless the solution makes it clear that this is not the case.)

Solution to Exercise 6

(a) We have

$$\begin{aligned}\operatorname{tr} \mathbf{A} &= 3 + 3 = 6, \\ \det \mathbf{A} &= 3 \times 3 - 2 \times 2 = 5.\end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 6\lambda + 5 = 0.$$

Hence

$$(\lambda - 1)(\lambda - 5) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 1 and 5.

The eigenvector equations have the form

$$\begin{pmatrix} 3 - \lambda & 2 \\ 2 & 3 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 1$, we obtain

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned}2x + 2y &= 0, \\ 2x + 2y &= 0.\end{aligned}$$

This pair of equations is equivalent to the single equation

$$x + y = 0, \quad \text{that is,} \quad y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1.

(Check:

$$\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.)$$

When $\lambda = 5$, the eigenvector equation is

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-2x + 2y = 0,$$

$$2x - 2y = 0.$$

This pair of equations is equivalent to the single equation

$$x - y = 0, \quad \text{that is, } y = x.$$

If $x = 1$, then $y = 1$. Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 5.

$$(\text{Check: } \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.)$$

(b) We have

$$\text{tr } \mathbf{A} = 3 + (-10) = -7,$$

$$\det \mathbf{A} = 3 \times (-10) - 2 \times (-6) = -18.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 7\lambda - 18 = 0.$$

Hence

$$(\lambda + 9)(\lambda - 2) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -9 and 2 .

The eigenvector equations have the form

$$\begin{pmatrix} 3 - \lambda & 2 \\ -6 & -10 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -9$, we obtain

$$\begin{pmatrix} 12 & 2 \\ -6 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$12x + 2y = 0,$$

$$-6x - y = 0.$$

This pair of equations is equivalent to the single equation

$$6x + y = 0, \quad \text{that is, } y = -6x.$$

If $x = 1$, then $y = -6$. Hence $\begin{pmatrix} 1 \\ -6 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -9 .

(Check:

$$\begin{pmatrix} 3 & 2 \\ -6 & -10 \end{pmatrix} \begin{pmatrix} 1 \\ -6 \end{pmatrix} = \begin{pmatrix} -9 \\ 54 \end{pmatrix} = -9 \begin{pmatrix} 1 \\ -6 \end{pmatrix}.)$$

When $\lambda = 2$, the eigenvector equation is

$$\begin{pmatrix} 1 & 2 \\ -6 & -12 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$x + 2y = 0,$$

$$-6x - 12y = 0.$$

This pair of equations is equivalent to the single equation

$$x + 2y = 0, \quad \text{that is, } x = -2y.$$

If $y = 1$, then $x = -2$. Hence $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

(Check:

$$\begin{pmatrix} 3 & 2 \\ -6 & -10 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.)$$

(c) We have

$$\text{tr } \mathbf{A} = 3 + 3 = 6,$$

$$\det \mathbf{A} = 3 \times 3 - 2 \times 0 = 9.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 6\lambda + 9 = 0.$$

Hence

$$(\lambda - 3)^2 = 0,$$

so $\mathbf{A}$ has a single, repeated eigenvalue 3.

The eigenvector equations have the form

$$\begin{pmatrix} 3 - \lambda & 2 \\ 0 & 3 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Substituting $\lambda = 3$ gives

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$0x + 2y = 0,$$

$$0x + 0y = 0.$$

This pair of equations is equivalent to the single equation

$$y = 0.$$

Exercise Booklet 11

Choose $x = 1$. Then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 3.

(Check: $\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.)

(d) We have

$$\mathbf{A}\mathbf{x} = 4\mathbf{x}$$

for every vector $\mathbf{x}$. So $\mathbf{A}$ has only a single, repeated eigenvalue 4, and any non-zero vector, such as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, is an eigenvector of $\mathbf{A}$ corresponding to this eigenvalue.

Solution to Exercise 7

We have

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 6 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix},$$

$$\begin{pmatrix} 4 & 0 & 2 \\ 0 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \\ 12 \end{pmatrix} = 6 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

So

$$\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

are eigenvectors of the matrix, and the corresponding eigenvalues are 3, 0 and 6, respectively.

Solution to Exercise 8

(a) Let λ be an eigenvalue of $\mathbf{A}$ with corresponding eigenvector $\mathbf{x}$. That is, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, where $\mathbf{x} \neq \mathbf{0}$. We will use mathematical induction to prove that λ^n is an eigenvalue of $\mathbf{A}^n$ with corresponding eigenvector $\mathbf{x}$, for all $n \in \mathbb{N}$. That is, we will prove that $\mathbf{A}^n\mathbf{x} = \lambda^n\mathbf{x}$, for all $n \in \mathbb{N}$.

Let $P(n)$ be the statement

$$\mathbf{A}^n\mathbf{x} = \lambda^n\mathbf{x}.$$

Then $P(1)$ is true, as

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}.$$

Now let $k \geq 1$, and assume that $P(k)$ is true, that is, $\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{x}$. We want to deduce that $P(k+1)$ is true, that is,

$$\mathbf{A}^{k+1}\mathbf{x} = \lambda^{k+1}\mathbf{x}.$$

Now,

$$\begin{aligned} \mathbf{A}^{k+1}\mathbf{x} &= \mathbf{A}(\mathbf{A}^k\mathbf{x}) \\ &= \mathbf{A}(\lambda^k\mathbf{x}), \quad \text{by } P(k), \\ &= \lambda^k(\mathbf{A}\mathbf{x}) \\ &= \lambda^k(\lambda\mathbf{x}) \\ &= \lambda^{k+1}\mathbf{x}, \end{aligned}$$

as required.

Thus

$$P(k) \Rightarrow P(k+1), \quad \text{for } k \geq 1.$$

Hence, by mathematical induction, $P(n)$ is true, for $n \geq 1$.

That is, λ^n is an eigenvalue of $\mathbf{A}^n$ with corresponding eigenvector $\mathbf{x}$, for all $n \in \mathbb{N}$.

(b) (i) Assume that λ is an eigenvalue of $\mathbf{A}$. Then $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for some non-zero vector $\mathbf{x}$. Hence

$$(-\mathbf{A})\mathbf{x} = -\mathbf{A}\mathbf{x} = -\lambda\mathbf{x}.$$

Since $\mathbf{x} \neq \mathbf{0}$, it follows that $-\lambda$ is an eigenvalue of $-\mathbf{A}$, with corresponding eigenvector $\mathbf{x}$.

(ii) Assume that λ is an eigenvalue of $\mathbf{A}$. Then $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for some non-zero vector $\mathbf{x}$. Hence

$$\begin{aligned} (\mathbf{A} + \mathbf{I})\mathbf{x} &= \mathbf{A}\mathbf{x} + \mathbf{I}\mathbf{x} = \lambda\mathbf{x} + \mathbf{x} \\ &= (\lambda + 1)\mathbf{x}. \end{aligned}$$

Since $\mathbf{x} \neq \mathbf{0}$, it follows that $\lambda + 1$ is an eigenvalue of $\mathbf{A} + \mathbf{I}$, with corresponding eigenvector $\mathbf{x}$.

(c) Assume that $\mathbf{A}^2 = \mathbf{A}$ and that λ is an eigenvalue of $\mathbf{A}$. Then $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ for some non-zero vector $\mathbf{x}$. Hence

$$\mathbf{A}^2\mathbf{x} = \mathbf{A}(\mathbf{A}\mathbf{x}) = \mathbf{A}(\lambda\mathbf{x}) = \lambda\mathbf{A}\mathbf{x} = \lambda^2\mathbf{x}.$$

Therefore, as $\mathbf{A}^2 = \mathbf{A}$,

$$\lambda^2\mathbf{x} = \lambda\mathbf{x}.$$

This can be rearranged as

$$\lambda^2\mathbf{x} - \lambda\mathbf{x} = \mathbf{0},$$

so

$$(\lambda^2 - \lambda)\mathbf{x} = \mathbf{0}.$$

As $\mathbf{x} \neq \mathbf{0}$, this gives

$$\lambda^2 - \lambda = 0,$$

that is,

$$\lambda(\lambda - 1) = 0.$$

Hence $\lambda = 0$ or $\lambda = 1$.

- (d) Assume that $\mathbf{A}$ is invertible and λ is an eigenvalue of $\mathbf{A}$, so $\mathbf{Ax} = \lambda\mathbf{x}$ for some non-zero vector $\mathbf{x}$. We will use proof by contradiction to show that $\lambda \neq 0$.

Assume that it is not true that $\lambda \neq 0$, so $\lambda = 0$. Then

$$\mathbf{Ax} = \lambda\mathbf{x} = 0\mathbf{x} = \mathbf{0},$$

that is,

$$\mathbf{Ax} = \mathbf{0}.$$

As $\mathbf{A}$ is invertible, we can multiply both sides of this equation on the left by $\mathbf{A}^{-1}$, to obtain

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{A}^{-1}\mathbf{0}.$$

As $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, the identity matrix, this gives

$$\mathbf{Ix} = \mathbf{0},$$

that is,

$$\mathbf{x} = \mathbf{0}.$$

This contradicts the fact that $\mathbf{x} \neq \mathbf{0}$.

Thus our assumption that $\lambda = 0$ is false, so $\lambda \neq 0$, as required.

Solution to Exercise 9

- (a) Since the matrix is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely -2 and 2 .

The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 2 .

The eigenvector equations have the form

$$\begin{pmatrix} -2 - \lambda & 0 \\ 4 & 2 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -2$, we obtain

$$\begin{pmatrix} 0 & 0 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$0x + 0y = 0,$$

$$4x + 4y = 0.$$

This pair of equations is equivalent to the single equation

$$x + y = 0, \quad \text{that is,} \quad y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue -2 .

(Check:

$$\begin{pmatrix} -2 & 0 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.)$$

- (b) Since the matrix is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely 4 and -3 .

The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue -3 .

The eigenvector equations have the form

$$\begin{pmatrix} 4 - \lambda & 0 \\ 2 & -3 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 4$, we obtain

$$\begin{pmatrix} 0 & 0 \\ 2 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$0x + 0y = 0,$$

$$2x - 7y = 0.$$

This pair of equations is equivalent to the single equation

$$2x - 7y = 0, \quad \text{that is,} \quad 7y = 2x.$$

If $x = 7$, then $y = 2$. Hence $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 4 .

(Check:

$$\begin{pmatrix} 4 & 0 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 28 \\ 8 \end{pmatrix} = 4 \begin{pmatrix} 7 \\ 2 \end{pmatrix}.)$$

- (c) Since the matrix is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely 1 and 2 .

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 1 .

The eigenvector equations have the form

$$\begin{pmatrix} 1 - \lambda & 3 \\ 0 & 2 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 2$, we obtain

$$\begin{pmatrix} -1 & 3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-x + 3y = 0,$$

$$0x + 0y = 0.$$

Exercise Booklet 11

This pair of equations is equivalent to the single equation

$$-x + 3y = 0, \quad \text{that is, } x = 3y.$$

If $y = 1$, then $x = 3$. Hence $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 2.

$$(\text{Check: } \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}.)$$

- (d) Since the matrix is a diagonal matrix, the eigenvalues are the elements on the leading diagonal, namely 4 and -2 .

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 4, and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue -2 .

Solution to Exercise 10

Since the matrix is a triangular matrix, the eigenvalues are the elements on the leading diagonal, so there is only a single, repeated eigenvalue 1.

The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix corresponding to the eigenvalue 1.

Solution to Exercise 11

- (a) We have

$$\text{tr } \mathbf{A} = 2 + 2 = 4,$$

$$\det \mathbf{A} = 2 \times 2 - 1 \times 4 = 0.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 4\lambda = 0.$$

Hence

$$\lambda(\lambda - 4) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 0 and 4.

The eigenvector equations have the form

$$\begin{pmatrix} 2 - \lambda & 1 \\ 4 & 2 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 0$, we obtain

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$2x + y = 0,$$

$$4x + 2y = 0.$$

This pair of equations is equivalent to the single equation

$$2x + y = 0, \quad \text{that is, } y = -2x.$$

If $x = 1$, then $y = -2$. Hence $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 0.

(Check:

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -2 \end{pmatrix}.)$$

When $\lambda = 4$, the eigenvector equation is

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-2x + y = 0,$$

$$4x - 2y = 0.$$

This pair of equations is equivalent to the single equation

$$-2x + y = 0, \quad \text{that is, } y = 2x.$$

If $x = 1$, then $y = 2$. Hence $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 4.

$$(\text{Check: } \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.)$$

(Here is an alternative way to find the eigenvalues and one of the eigenvectors.

Since $\det \mathbf{A} = 0$, the matrix represents a flattening, so the eigenvalues are 0 and $\text{tr } \mathbf{A}$, and here $\text{tr } \mathbf{A} = 4$. It also follows that the columns of $\mathbf{A}$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue

$\text{tr } \mathbf{A} = 4$. The eigenvector $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, obtained from the first column, is a scalar multiple of the eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, obtained from the second column. Both vectors are acceptable answers.)

- (b) We have

$$\text{tr } \mathbf{A} = 3 + 3 = 6,$$

$$\det \mathbf{A} = 3 \times 3 - 3 \times 3 = 0.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 6\lambda = 0.$$

This gives

$$\lambda(\lambda - 6) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 0 and 6.

The eigenvector equations have the form

$$\begin{pmatrix} 3-\lambda & 3 \\ 3 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 0$, we obtain

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$3x + 3y = 0,$$

$$3x + 3y = 0.$$

This pair of equations is equivalent to the single equation

$$x + y = 0, \quad \text{that is, } y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 0.

(Check:

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.)$$

When $\lambda = 6$, the eigenvector equation is

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-3x + 3y = 0,$$

$$3x - 3y = 0.$$

This pair of equations is equivalent to the single equation

$$x - y = 0, \quad \text{that is, } y = x.$$

If $x = 1$, then $y = 1$. Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 6.

$$(\text{Check: } \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.)$$

(Alternatively, you can find the eigenvalues and one of the eigenvectors by using a similar strategy to that shown at the end of the solution to part (a).)

Solution to Exercise 12

We have

$$\text{tr } \mathbf{A} = 0 + 0 = 0,$$

$$\det \mathbf{A} = 0 \times 0 - 1 \times (-1) = 1.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 1 = 0.$$

This gives

$$\lambda^2 = -1,$$

so the eigenvalues of $\mathbf{A}$ are $-i$ and i .

(Check: $(-i) + i = 0 = \text{tr } \mathbf{A}$.)

Solution to Exercise 13

(a) This matrix represents a reflection in a line through the origin because it is of the form

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix},$$

with $\alpha = 3\pi/4$.

Therefore it has eigenvalues -1 and 1 , and corresponding eigenvectors

$$\begin{pmatrix} -\sin(3\pi/4) \\ \cos(3\pi/4) \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

and

$$\begin{pmatrix} \cos(3\pi/4) \\ \sin(3\pi/4) \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},$$

respectively. Multiplying these eigenvectors by $-\sqrt{2}$ and $\sqrt{2}$, respectively, gives the simpler pair of eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(Check:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.)$$

Exercise Booklet 11

- (b) This matrix represents a reflection in a line through the origin because it is of the form

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix},$$

with $\alpha = \pi/3$.

Therefore it has eigenvalues -1 and 1 , and corresponding eigenvectors

$$\begin{pmatrix} -\sin(\pi/3) \\ \cos(\pi/3) \end{pmatrix} = \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

and

$$\begin{pmatrix} \cos(\pi/3) \\ \sin(\pi/3) \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix},$$

respectively. Multiplying each of these eigenvectors by 2 gives the simpler pair of eigenvectors

$$\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix},$$

respectively.

(Check:

$$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ -1 \end{pmatrix} \\ = - \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \\ = 1 \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}.)$$

Solution to Exercise 14

- (a) Since $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 5, we have

$$\begin{aligned} \mathbf{A} \begin{pmatrix} 3 \\ 3 \end{pmatrix} &= \mathbf{A} \times 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 3\mathbf{A} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 3 \times 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 15 \\ 15 \end{pmatrix}. \end{aligned}$$

So f maps the point $(3, 3)$ to the point $(15, 15)$.

- (b) Since $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1, we have

$$\begin{aligned} \mathbf{A} \begin{pmatrix} -2 \\ 2 \end{pmatrix} &= \mathbf{A}(-2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= -2\mathbf{A} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= -2 \times 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 2 \end{pmatrix}. \end{aligned}$$

So f maps the point $(-2, 2)$ to the point $(-2, 2)$.

- (c) Since $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$ corresponding to the eigenvalues 1 and 5, respectively, we have (using the hint)

$$\begin{aligned} \mathbf{A} \begin{pmatrix} 7 \\ 1 \end{pmatrix} &= \mathbf{A} \left(3 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \\ &= 3\mathbf{A} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4\mathbf{A} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 3 \times 1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 4 \times 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 23 \\ 17 \end{pmatrix}. \end{aligned}$$

So f maps the point $(7, 1)$ to the point $(23, 17)$.

Solution to Exercise 15

- (a) Since $\mathbf{A}$ is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely 2 and 3.

The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 3.

The eigenvector equations have the form

$$\begin{pmatrix} 2 - \lambda & 0 \\ 5 & 3 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 2$, we obtain

$$\begin{pmatrix} 0 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned} 0x + 0y &= 0, \\ 5x + y &= 0. \end{aligned}$$

This pair of equations is equivalent to the single equation

$$5x + y = 0, \quad \text{that is, } y = -5x.$$

If $x = 1$, then $y = -5$. Hence $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

Let

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix}.$$

Then

$$\det \mathbf{P} = 0 \times (-5) - 1 \times 1 = -1,$$

so

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{pmatrix} -5 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

(Check:

$$\begin{aligned} & \begin{pmatrix} 0 & 1 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 3 & -10 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}. \end{aligned}$$

(b) We have

$$\operatorname{tr} \mathbf{A} = 2 + 3 = 5,$$

$$\det \mathbf{A} = 2 \times 3 - 6 \times 2 = -6.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 5\lambda - 6 = 0.$$

This gives

$$(\lambda + 1)(\lambda - 6) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -1 and 6 .

The eigenvector equations have the form

$$\begin{pmatrix} 2 - \lambda & 6 \\ 2 & 3 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -1$, we obtain

$$\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$3x + 6y = 0,$$

$$2x + 4y = 0.$$

This pair of equations is equivalent to the single equation

$$x + 2y = 0, \quad \text{that is, } x = -2y.$$

If $y = 1$, then $x = -2$. Hence $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -1 .

When $\lambda = 6$, the eigenvector equation is

$$\begin{pmatrix} -4 & 6 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-4x + 6y = 0,$$

$$2x - 3y = 0.$$

This pair of equations is equivalent to the single equation

$$2x - 3y = 0, \quad \text{that is, } 3y = 2x.$$

If $x = 3$, then $y = 2$. Hence $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 6.

Let

$$\mathbf{P} = \begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix}.$$

Then

$$\det \mathbf{P} = -2 \times 2 - 3 \times 1 = -7,$$

so

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{pmatrix} 2 & -3 \\ -1 & -2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix}.$$

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}.$$

(Check:

$$\begin{aligned} & \begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix} \frac{1}{7} \begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{7} \begin{pmatrix} 2 & 18 \\ -1 & 12 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{7} \begin{pmatrix} 14 & 42 \\ 14 & 21 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix}. \end{aligned}$$

Exercise Booklet 11

(c) We have

$$\begin{aligned}\operatorname{tr} \mathbf{A} &= (-1) + (-3) = -4, \\ \det \mathbf{A} &= (-1) \times (-3) - 4 \times 2 = -5.\end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 4\lambda - 5 = 0.$$

This gives

$$(\lambda + 5)(\lambda - 1) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -5 and 1 .

The eigenvector equations have the form

$$\begin{pmatrix} -1 - \lambda & 4 \\ 2 & -3 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -5$, we obtain

$$\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$4x + 4y = 0,$$

$$2x + 2y = 0.$$

This pair of equations is equivalent to the single equation

$$x + y = 0, \quad \text{that is, } y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -5 .

When $\lambda = 1$, the eigenvector equation is

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-2x + 4y = 0,$$

$$2x - 4y = 0.$$

This pair of equations is equivalent to the single equation

$$x - 2y = 0, \quad \text{that is, } x = 2y.$$

If $y = 1$, then $x = 2$. Hence $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 1 .

Let

$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\det \mathbf{P} = 1 \times 1 - 2 \times (-1) = 3,$$

so

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}.$$

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Check:

$$\begin{aligned}& \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \\&= \frac{1}{3} \begin{pmatrix} -5 & 2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \\&= \frac{1}{3} \begin{pmatrix} -3 & 12 \\ 6 & -9 \end{pmatrix} \\&= \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}.)\end{aligned}$$

(In this exercise and in other exercises that require you to diagonalise a matrix, you may have chosen to order the eigenvectors that form matrix $\mathbf{P}$ and the corresponding eigenvalues in matrix $\mathbf{D}$ in the opposite way to that given in the solution. You may also have chosen different eigenvectors. Your eigenvectors should be scalar multiples of those given in the solution.)

Solution to Exercise 16

- (a) $\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}^3 = \begin{pmatrix} 4^3 & 0 \\ 0 & (-1)^3 \end{pmatrix} = \begin{pmatrix} 64 & 0 \\ 0 & -1 \end{pmatrix}$
- (b) $\begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}^4 = \begin{pmatrix} (-3)^4 & 0 \\ 0 & 2^4 \end{pmatrix} = \begin{pmatrix} 81 & 0 \\ 0 & 16 \end{pmatrix}$
- (c) $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}^{10} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 0^{10} \end{pmatrix} = \begin{pmatrix} 1024 & 0 \\ 0 & 0 \end{pmatrix}$

Solution to Exercise 17

- (a) Since $\mathbf{A}$ is a triangular matrix, the eigenvalues are the elements on the leading diagonal, namely 2 and -2 .

The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -2 .

The eigenvector equations have the form

$$\begin{pmatrix} 2 - \lambda & 0 \\ 1 & -2 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 2$, we obtain

$$\begin{pmatrix} 0 & 0 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned} 0x + 0y &= 0, \\ x - 4y &= 0. \end{aligned}$$

This pair of equations is equivalent to the single equation

$$x - 4y = 0, \quad \text{that is, } x = 4y.$$

If $y = 1$, then $x = 4$. Hence $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

Let

$$\mathbf{P} = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\det \mathbf{P} = 0 \times 1 - 4 \times 1 = -4,$$

so

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{pmatrix} 1 & -4 \\ -1 & 0 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -1 & 4 \\ 1 & 0 \end{pmatrix}.$$

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

(Check:

$$\begin{aligned} & \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{4} \begin{pmatrix} -1 & 4 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 0 & 8 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 1 & 0 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 8 & 0 \\ 4 & -8 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix}. \end{aligned}$$

Then

$$\mathbf{D}^6 = \begin{pmatrix} (-2)^6 & 0 \\ 0 & 2^6 \end{pmatrix} = \begin{pmatrix} 64 & 0 \\ 0 & 64 \end{pmatrix}.$$

So

$$\begin{aligned} \mathbf{A}^6 &= \mathbf{PD}^6\mathbf{P}^{-1} \\ &= \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 64 & 0 \\ 0 & 64 \end{pmatrix} \frac{1}{4} \begin{pmatrix} -1 & 4 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 64 \\ 16 & 16 \end{pmatrix} \begin{pmatrix} -1 & 4 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 64 & 0 \\ 0 & 64 \end{pmatrix}. \end{aligned}$$

(Check: $\text{tr}(\mathbf{A}^6) = 64 + 64 = 128$
and $\text{tr}(\mathbf{D}^6) = 64 + 64 = 128$.)

(b) We have

$$\begin{aligned} \text{tr } \mathbf{A} &= 5 + (-7) = -2, \\ \det \mathbf{A} &= 5 \times (-7) - 9 \times (-3) = -8. \end{aligned}$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + 2\lambda - 8 = 0.$$

This gives

$$(\lambda + 4)(\lambda - 2) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -4 and 2 .

The eigenvector equations have the form

$$\begin{pmatrix} 5 - \lambda & 9 \\ -3 & -7 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -4$, we obtain

$$\begin{pmatrix} 9 & 9 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned} 9x + 9y &= 0, \\ -3x - 3y &= 0. \end{aligned}$$

This pair of equations is equivalent to the single equation

$$x + y = 0, \quad \text{that is, } y = -x.$$

If $x = 1$, then $y = -1$. Hence $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -4 .

When $\lambda = 2$, the eigenvector equation is

$$\begin{pmatrix} 3 & 9 \\ -3 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned} 3x + 9y &= 0, \\ -3x - 9y &= 0. \end{aligned}$$

This pair of equations is equivalent to the single equation

$$x + 3y = 0, \quad \text{that is, } x = -3y.$$

If $y = 1$, then $x = -3$. Hence $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

Let

$$\mathbf{P} = \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix}.$$

Then

$$\det \mathbf{P} = 1 \times 1 - (-3) \times (-1) = -2,$$

so

$$\mathbf{P}^{-1} = \frac{1}{\det \mathbf{P}} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}.$$

We have

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where

$$\mathbf{D} = \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix}.$$

(Check:

$$\begin{aligned} & \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix} \left(-\frac{1}{2} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 9 \\ -3 & -7 \end{pmatrix}. \end{aligned}$$

Then

$$\mathbf{D}^3 = \begin{pmatrix} (-4)^3 & 0 \\ 0 & 2^3 \end{pmatrix} = \begin{pmatrix} -64 & 0 \\ 0 & 8 \end{pmatrix}.$$

So

$$\begin{aligned} \mathbf{A}^3 &= \mathbf{PD}^3\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -64 & 0 \\ 0 & 8 \end{pmatrix} \left(-\frac{1}{2} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 32 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 32 & 12 \\ -32 & -4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 44 & 108 \\ -36 & -100 \end{pmatrix}. \end{aligned}$$

(Check: $\text{tr}(\mathbf{A}^3) = 44 + (-100) = -56$
and $\text{tr}(\mathbf{D}^3) = (-64) + 8 = -56$.)

Solution to Exercise 18

$$\text{We have } \mathbf{x}_0 = \begin{pmatrix} 500 \\ 2000 \end{pmatrix} = 500 \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Now

$$\mathbf{P}^{-1} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -0.2 & 0.4 \\ 0.6 & -0.2 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1.4 \\ -0.2 \end{pmatrix}.$$

So

$$\begin{aligned} \mathbf{D}^{10}\mathbf{P}^{-1} \begin{pmatrix} 1 \\ 4 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0.95^{10} \end{pmatrix} \begin{pmatrix} 1.4 \\ -0.2 \end{pmatrix} \\ &= \begin{pmatrix} 1.4 \\ -0.2 \times 0.95^{10} \end{pmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{A}^{10} \begin{pmatrix} 1 \\ 4 \end{pmatrix} &= \mathbf{PD}^{10}\mathbf{P}^{-1} \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1.4 \\ -0.2 \times 0.95^{10} \end{pmatrix} \\ &= \begin{pmatrix} 1.4 - 0.4 \times 0.95^{10} \\ 4.2 - 0.2 \times 0.95^{10} \end{pmatrix}. \end{aligned}$$

So

$$\begin{aligned} \mathbf{A}^{10}\mathbf{x}_0 &= \mathbf{A}^{10} \begin{pmatrix} 500 \\ 2000 \end{pmatrix} \\ &= 500\mathbf{A}^{10} \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= 500 \begin{pmatrix} 1.4 - 0.4 \times 0.95^{10} \\ 4.2 - 0.2 \times 0.95^{10} \end{pmatrix} \\ &= \begin{pmatrix} 700 - 200 \times 0.95^{10} \\ 2100 - 100 \times 0.95^{10} \end{pmatrix} \\ &= \begin{pmatrix} 580.25 \dots \\ 2040.12 \dots \end{pmatrix}. \end{aligned}$$

The model predicts that, to the nearest integers, there will be 580 wolves and 2040 hares after 10 years.

Solution to Exercise 19

$$(a) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$(b) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 10 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution to Exercise 20

(a) This is a decoupled system of differential equations, so the general solution is

$$x = Ce^{5t}, \quad y = De^{-4t},$$

where C and D are arbitrary constants.

- (b) This is a decoupled system of differential equations, so the general solution is

$$x = Ce^{-0.1t}, \quad y = De^{0.6t},$$

where C and D are arbitrary constants.

Solution to Exercise 21

- (a) The matrix form of the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}.$$

We have

$$\text{tr } \mathbf{A} = 3 + 4 = 7,$$

$$\det \mathbf{A} = 3 \times 4 - 2 \times 1 = 10.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 - 7\lambda + 10 = 0.$$

This gives

$$(\lambda - 2)(\lambda - 5) = 0,$$

so the eigenvalues of $\mathbf{A}$ are 2 and 5.

The eigenvector equations have the form

$$\begin{pmatrix} 3 - \lambda & 2 \\ 1 & 4 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = 2$, we obtain

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$x + 2y = 0,$$

$$x + 2y = 0.$$

This pair of equations is equivalent to the single equation

$$x + 2y = 0, \quad \text{that is, } x = -2y.$$

If $y = 1$, then $x = -2$. Hence $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2.

(Check:

$$\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.)$$

When $\lambda = 5$, the eigenvector equation is

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-2x + 2y = 0,$$

$$x - y = 0.$$

This pair of equations is equivalent to the single equation

$$x - y = 0, \quad \text{that is, } y = x.$$

If $x = 1$, then $y = 1$. Hence $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 5.

$$(\text{Check: } \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.)$$

The general solution is

$$\mathbf{x} = Ce^{2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + De^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where C and D are arbitrary constants. Hence

$$x = -2Ce^{2t} + De^{5t},$$

$$y = Ce^{2t} + De^{5t}.$$

- (b) The matrix form of the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} -13 & 5 \\ -30 & 12 \end{pmatrix}.$$

We have

$$\text{tr } \mathbf{A} = (-13) + 12 = -1,$$

$$\det \mathbf{A} = (-13) \times 12 - 5 \times (-30) = -6.$$

So the characteristic equation of $\mathbf{A}$ is

$$\lambda^2 + \lambda - 6 = 0.$$

This gives

$$(\lambda + 3)(\lambda - 2) = 0,$$

so the eigenvalues of $\mathbf{A}$ are -3 and 2 .

The eigenvector equations have the form

$$\begin{pmatrix} -13 - \lambda & 5 \\ -30 & 12 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

When $\lambda = -3$, we obtain

$$\begin{pmatrix} -10 & 5 \\ -30 & 15 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$-10x + 5y = 0,$$

$$-30x + 15y = 0.$$

Exercise Booklet 11

This pair of equations is equivalent to the single equation

$$2x - y = 0, \quad \text{that is, } y = 2x.$$

If $x = 1$, then $y = 2$. Hence $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue -3 .

(Check:

$$\begin{pmatrix} -13 & 5 \\ -30 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ -6 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.)$$

When $\lambda = 2$, the eigenvector equation is

$$\begin{pmatrix} -15 & 5 \\ -30 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives

$$\begin{aligned} -15x + 5y &= 0, \\ -30x + 10y &= 0. \end{aligned}$$

This pair of equations is equivalent to the single equation

$$3x - y = 0, \quad \text{that is, } y = 3x.$$

If $x = 1$, then $y = 3$. Hence $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 2 .

(Check:

$$\begin{pmatrix} -13 & 5 \\ -30 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}.)$$

The general solution is

$$\mathbf{x} = Ce^{-3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + De^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$

where C and D are arbitrary constants.
Hence

$$\begin{aligned} x &= Ce^{-3t} + De^{2t}, \\ y &= 2Ce^{-3t} + 3De^{2t}. \end{aligned}$$